Lecture Notes Dated: Jan 18, 2013

Deduction from Planck's Law

From this Planck's radiation formula, it is possible to derive the other important laws of the theory of radiation. This is done below.

Stefan-Boltzmann Law

In the theory of radiation Stefan's law states that, *the rate of emission of radiant energy by unit area of perfectly blackbody is directly proportional to the forth power of its absolute temperature*, *i.e.*,

$$E=\sigma T^4 \qquad (15)$$

where σ is a constant and is called Stefan's constant. This law simply deals with the amount of heat emitted by the body by virtue of its temperature irrespective of what it receives from the surroundings.

The law can be extended to represent the net loss of heat and may be completed as follows. If a blackbody at absolute temperature T be surrounded by another blackbody at absolute temperature T_0, then the inner blackbody not only loses an amount of energy σT^4 but also gains σT_0^4, thus the amount of energy E lost per second per unit area of the former is,

$$E=\sigma(T^4-T_0^4) \qquad (16)$$

Derivation of Stefan's Law from Planck's Law

As we know from definitions, that the total energy density u of the radiation in a cavity is the integral of the energy density over all frequencies, *i.e.*,

$$u=\int_0^\infty u(\nu)\,d\nu$$

From Planck's radiation formula (eq. 13),

$$u=\frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3\,d\nu}{e^{h\nu/kT}-1}$$

Let us make a substitution, $x=\frac{h\nu}{kT}$, so that $\nu=\frac{kT}{h}x$ and then $d\nu=\frac{kT}{h}dx$

Hence, we have $u=\frac{8\pi h}{c^3}\int_0^\infty\left(\frac{kT}{h}\right)^3\frac{x^3}{e^x-1}\left(\frac{kT}{h}\right)dx$

$$u=\frac{8\pi k^4T^4}{h^3c^3}\int_0^\infty\frac{x^3}{e^x-1}dx$$

$$u=\frac{8\pi k^4T^4}{h^3c^3}\int_0^\infty x^3\,dx\,(e^{-x}+e^{-2}x+\ldots+e^{-nx}+\ldots)$$

$$u=\frac{48\pi k^4T^4}{h^3c^3}.\frac{\pi^4}{90}=\left(\frac{8\pi^5k^4}{15c^3h^3}\right)T^4$$

because $\int_0^\infty x^3e^{-nx}dx=\frac{6}{n^4}$ and $\sum_{n=1}^{n=\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$.

Hence, $u=aT^4$

where $a=\frac{8\pi^5 k^4}{15 c^3 h^3}$ is a universal constant.

The total energy density, u is proportional to the forth power of the absolute temperature of the cavity walls. We therefore expect that the energy radiated by an object per unit time per unit area, E is also proportional to T^4, a conclusion well known as ***Stefan-Boltzmann law***.

$$E=e\sigma T^4$$

The emissivity e depends on the nature of the radiating surface and ranges from 0 (for a perfect reflector, which does not radiate at all) to 1 (for blackbody). Thus, Stefan-Boltzmann law for a perfect blackbody reads

$$E=\sigma T^4$$

This is the same expression obtained from thermodynamical cosideration.

Stefan Constant

Since $\sigma=ac/4$, we have

$$\sigma=\frac{2}{15}\frac{\pi^5 k^4}{c^2 h^3}=5.67\times10^{-8}\frac{joule}{m^{2.} s. K^4} \tag{17}$$

Wien's Displacement Law

An interesting feature of the blackbody spectrum at a given temperature is the wavelength λ_{max} for which the energy density is the greatest. According to Wein's law

$$\lambda_{max} T=constant=b(say) \tag{18}$$

Derivation of Wien's Displacement Law from Planck's Law

To find λ_{max} we use Planck's radiation formula in terms of wavelength (eq. 14). for maximum emission of energy, i.e., for maximum value of $u(\lambda)$, $du(\lambda)/d\lambda$ must be equal to zero for $\lambda = \lambda_{max}$, *i.e.*,

$$\frac{du(\lambda)}{d\lambda}=0$$

$$8\pi c h\left[\frac{-5}{\lambda^6}\cdot\frac{1}{e^{hc/\lambda kT}-1}+\frac{1}{\lambda^5}\cdot\frac{(hc/\lambda^2 kT)e^{hc/\lambda kT}}{(e^{hc/\lambda kT}-1)^2}\right]=0$$

$$\frac{8\pi c h}{\lambda^6(e^{hc/\lambda kT}-1)}\left[-5+\frac{hc}{\lambda kT}\frac{e^{hc/\lambda kT}}{(e^{hc/\lambda kT}-1)}\right]=0$$

Hence $$-5+\frac{hc}{\lambda kT}\frac{e^{hc/\lambda kT}}{(e^{hc/\lambda kT}-1)}=0$$

Putting $hc/\lambda kT = x$, we get

$$-5+\frac{x e^x}{(e^x-1)}=0$$

or, $$x e^x-5 e^x+5=0$$

Applying the method of approximation the exact value of x is found to be equal to 4.965. This represents the wavelength at which the energy density is maximum, i.e., $\lambda = \lambda_{max}$.

$$\frac{hc}{kT\lambda_{max}}=4.965$$

which is more conveniently expressed as,

$$\lambda_{max}T=\frac{hc}{4.965\,k}=2.898\times10^{-3}\,m.K$$

This equation is known as ***Wien's displacement law***. It quantitatively expresses the empirical fact that the peak in the blackbody spectrum shifts to progressively shorter wavelengths (higher frequencies) as the temperature increased. It can also be expressed as

$$\frac{\nu_{max}}{T}=\frac{4.965\,k}{h}=1.035\times10^{8}\,sec^{-1.}\,K^{-1}$$

Wien's Constant

According to Wien's law

$$\lambda_{max}T=constant=b(say)$$

Hence, Wien's constant, b

$$b=\frac{hc}{4.965\,k}=2.898\times10^{-3}\,m.K$$

Deduction of Newton's Law of Cooling from Stefan's Law

The Stefan-Boltzmann law states that, if a blackbody at absolute temperature T be surrounded by another blackbody at absolute temperature T_0, then the inner blackbody not only loses an amount of energy σT^4 but also gains σT_0^4, thus the amount of energy E lost per second per unit area of the former is,

$$E=\sigma(T^4-T_0^4)$$

If the temperature T of a blackbody is not too great than the temperature T_0 of surrounding blackbody, then we can put T such that, $T=T_0+dT$. Then

$$E=\sigma[(T_0+dT)^4-T_0^4]$$

or,
$$E=\sigma[4T_0^3dT+6T_0^2dT^2+4T_0dT^3+dT^4]$$

As dT is small, therefore its higher order terms may be neglected. Thus

$$E=4\sigma T_0^3dT=4\sigma T_0^3(T-T_0)$$

or,
$$E\propto(T-T_0)$$

This results states that, for small differences of temperature the heat loss due to radiation is proportional to the temperature difference between the blackbody and the surroundings. This is actually the statement of ***Newton's law of cooling***.